MATH-4800
Numerical Computing

Summer 2023
Schwendeman/Traldi

Exam 1 Solutions

Thursday, June 29, 2023

Name (print clearly in the box):

RIN (print clearly in the box):

Instructions:

1. Please answer all 5 problems with sufficient supporting work.
2. No electronic gizmos (laptops, calculators, cell phones, etc.) permitted.
3. No books or notes permitted, but one crib sheet is allowed.
4. Write out and sign the **Honor Pledge** below.

Problem (pts)	Score
1 (12pts)	
2 (12pts)	
3 (12pts)	
4 (12pts)	
5 (12pts)	
Total (60pts)	

Honor Pledge: I have neither given nor received any illegal aid on this exam.

1. [12pts] (a) Let $S(x) = \sqrt{6-2x}$. An approximation of $S(x)$ is given by

$$\tilde{S}(x) = \frac{5}{2} - \frac{x}{2}$$

Compute (i) the absolute forward error in the approximation when $x = 5/2$ and (ii) the relative backward error in the approximation when $x = 3/2$.

Solution

The absolute forward error is

$$|\tilde{S}(5/2) - S(5/2)| = |5/4 - 1| = 1/4$$

The relative backward error is $|\tilde{x} - x|/|x|$, where $S(\tilde{x}) = \tilde{S}(x)$. Have

$$\tilde{S}(3/2) = \frac{7}{4} = S(\tilde{x}) = \sqrt{6 - 2\tilde{x}} \qquad \Rightarrow \qquad \tilde{x} = 47/32$$

Thus,

$$\frac{|\tilde{x} - x|}{|x|} = \frac{|47/32 - 3/2|}{|3/2|} = \frac{1}{48}$$

1. (b) Let $Q(x, y) = x + \sqrt{x^2 + y^2}$. The function Q is computed using finite precision arithmetic for the following (x, y) pairs: (i) $(10^{-5}, 10^5)$ and (ii) $(-10^5, 10^{-5})$. For which of the two pairs, if any or both, does the calculation of Q lead to a large loss of significant digits (i.e. more than 2 or 3 significant digits say)? Explain.

Solution

For the first pair, we have $x = 10^{-5}$ and $y = 10^5$. Since $x \ll y$, we have $\sqrt{x^2 + y^2} \approx |y| = 10^5$ with no loss in significant figures. Thus,

$$Q \approx x + |y| \approx |y| = 10^5$$

with no loss of significant figures.

For the second pair, we have $x = -10^5$ and $y = 10^{-5}$ and now $|x| \gg y$. Thus, $\sqrt{x^2 + y^2} \approx |x| = 10^5$ with no loss in significant figures. However,

$$Q \approx x + |x| \approx 0$$

with loss of significant figures with nearly equal numbers are subtracted.

2. [12pts] (a) Let

$$f(x) = \frac{x}{x+1}$$

and suppose $P(x)$ is the second-degree (quadratic) Taylor polynomial approximation of $f(x)$ about the point $x_0 = 1$. Use the remainder term to estimate $|P(1.1) - f(1.1)|$.

Solution

Use

$$f'(x) = \frac{1}{(x+1)^2}, \qquad f''(x) = -\frac{2}{(x+1)^3}, \qquad f'''(x) = \frac{6}{(x+1)^4}$$

and

$$f(1) = \frac{1}{2}, \qquad f'(1) = \frac{1}{4}, \qquad f''(1) = -\frac{1}{4}.$$

to get

$$P(x) = \frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2$$

The remainder term is

$$R(x) = \frac{f'''(c)}{3!}(x-1)^3 = \frac{1}{(c+1)^4}(x-1)^3$$

where c is between x and 1. For $x = 1.1$, use $c = 1$ to maximize the remainder. Thus,

$$|P(1.1) - f(1.1)| \leq \frac{1}{16}10^{-3}$$

2. (b) Let $f(x) = x^3 - x^2 - 3x + 1$. A root of f is bracketed on the interval $x \in [0, 1]$. Use **two** steps of the method of false position to determine an approximation for the root.

Solution

Note that $f(0) = 1$ and $f(1) = -2$. The line $L_1(x)$ through the points $(x, y) = (0, 1)$ and $(1, -2)$ is

$$L_1(x) = 1 + \frac{-2 - 1}{1 - 0} x = 1 - 3x$$

Set $L_1 = 0$ to give the first approximation $r_1 = 1/3$. Since $f(r_1) = -2/27$, the new bracket is $[0, 1/3]$. The line $L_2(x)$ through the points $(x, y) = (0, 1)$ and $(1/3, -2/27)$ is

$$L_2(x) = 1 + \frac{-2/27 - 1}{1/3 - 0} x = 1 - \frac{29}{9} x$$

The approximation given by two steps is r_2 , where $L_2(r_2) = 0$, which implies $r_2 = 9/27$.

3. [12pts] (a) Let

$$g(x) = 5 + \frac{2}{3}\sin x, \qquad I = [\pi, 2\pi]$$

Student X claims that the fixed-point iteration, $x_{n+1} = g(x_n)$, converges to the unique fixed point in I for *any* starting value x_0 in I. Use the Fixed-Point theorem to support this claim or state why the claim may not be true.

Solution

First show that g maps I into itself. Check endpoints

$$g(\pi) = g(2\pi) = 5 \in I$$

Also, check for local extrema

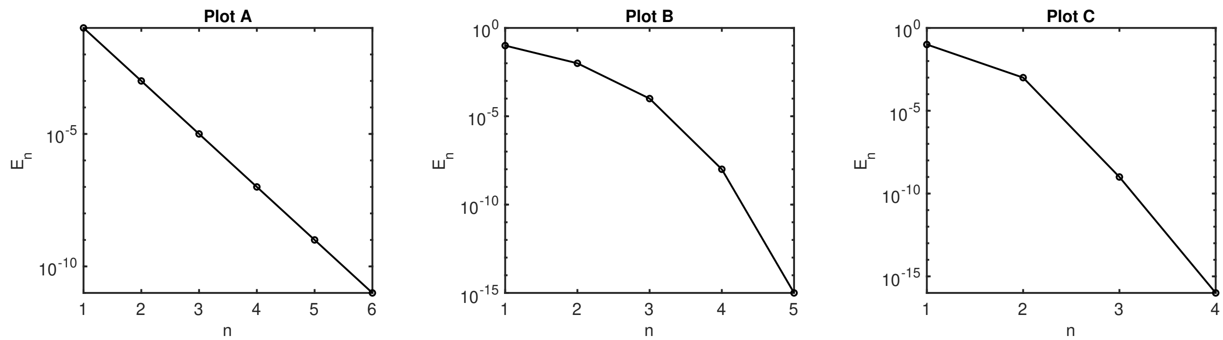
$$g'(x) = \frac{2}{3}\cos x = 0 \qquad \Rightarrow \qquad x = \frac{3\pi}{2} \in I$$

and

$$g(3\pi/2) = 5 - \frac{2}{3} \in I$$

Thus $g \in I$ whenever $x \in I$. Second, note that $|g'| \leq 2/3$ for all x including $x \in I$. Thus, according to the Fixed-Point Theorem, the iteration converges for all $x_0 \in I$. Student X is correct.

3. (b) A smooth function $F(x)$ has a simple root at $x = r$. Newton's method is used to compute the root using a starting value x_0 chosen close enough to r so that the iteration converges. Of the three plots below, which one best describes the behavior of the error, $E_n = |x_n - r|$, as a function of n. Explain.



Solution

Newton's method converges quadratically so that $E_{n+1} \approx CE_n^2$, where C is a constant. Consider the data...

For Plot A, the data appears to be

$$E_1 = 10^{-1}, \quad E_2 = 10^{-3}, \quad E_3 = 10^{-5}, \quad \dots \qquad \Rightarrow \qquad E_{n+1} \approx \tfrac{1}{100} E_n \text{, i.e. linear convergence}$$

For Plot B, the data appears to be

$$E_1 = 10^{-1}, \quad E_2 = 10^{-2}, \quad E_3 = 10^{-4}, \quad \dots \qquad \Rightarrow \qquad E_{n+1} \approx E_n^2 \text{, i.e. quadratic convergence}$$

Finally, the data for Plot C appears to be

$$E_1 = 10^{-1}, \quad E_2 = 10^{-3}, \quad E_3 = 10^{-9}, \quad \dots \qquad \Rightarrow \qquad E_{n+1} \approx E_n^3 \text{, i.e. cubic convergence}$$

The answer is Plot B.

4. [12pts] (a) Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 6 & 2 & 1 \\ -2 & 1 & -4 \end{bmatrix}$$

Use row elimination to find a lower triangular matrix L (with ones on the main diagonal) and an upper triangular matrix U such that $LU = A$.

Solution

Let

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad M_1 A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & -5 \end{bmatrix}$$

and then

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad \Rightarrow \qquad M_2 M_1 A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

So,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

4. (b) Consider the following sequence of Matlab commands:

```
>> n=100;  A=rand(n,n);
>> x=ones(n,1);  b=A*x;
>> xc=myLinearSystemSolver(A,b);
>> norm(b-A*xc)/norm(b)

ans =

   4.2804e-16

>> delta=norm(xc-x)/norm(x);
```

Determine an estimate for the value of `delta` or explain why there is insufficient information given to provide an accurate estimate.

Solution

Since

$$\frac{\|\mathbf{b} - A\mathbf{x}_c\|}{\|\mathbf{b}\|} = O(\epsilon_{\text{mach}})$$

suggests that the linear solver is backwards stable. Thus,

$$\texttt{delta} = \frac{\|\mathbf{x}_c - \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\epsilon_{\text{mach}}$$

Since the condition number $\kappa(A)$ is not given, there is insufficient information given to provide an accurate estimate.

5. [12pts] Let

$$A = \begin{bmatrix} \alpha & 2 & 0 \\ 2 & 4 & \beta \\ 0 & \beta & 5 \end{bmatrix}$$

where α and β are real constants.

(a) Determine all values of α such that A is symmetric, positive definite when $\beta = 0$.

(b) Find the Cholesky factorization of A when $\alpha = 3$ and $\beta = -2$.

Solutions

(a) Note that A is symmetric for all values of α and β. If $\beta = 0$, we have

$$\mathbf{x}^T A \mathbf{x} = \alpha x_1^2 + 4x_1x_2 + 4x_2^2 + 5x_3^2 = (\alpha - 1)x_1^2 + (x_1 + 2x_2)^2 + 5x_3^2 \geq 0 \qquad \text{if } \alpha \geq 1 .$$

Also, $\mathbf{x}^T A \mathbf{x} = 0$ implies $x_3 = 0$, $x_1 + 2x_2 = 0$ and $(\alpha - 1)x_1^2 = 0$. If $\alpha > 1$, then $x_1 = 0$ which implies $x_2 = 0$. So, A is symmetric, positive definite if $\alpha > 1$.

(a) Let

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ 0 & \ell_{32} & \ell_{33} \end{bmatrix} \qquad \Rightarrow \qquad LL^T = \begin{bmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} & 0 \\ \ell_{11}\ell_{21} & \ell_{22}^2 + \ell_{21}^2 & \ell_{22}\ell_{32} \\ 0 & \ell_{22}\ell_{32} & \ell_{33}^2 + \ell_{32}^2 \end{bmatrix}$$

Set $A = LL^T$ when $\alpha = 3$ and $\beta = -2$ gives

$$\ell_{11} = \sqrt{3}, \qquad \ell_{21} = 2/\sqrt{3}, \qquad \ell_{22} = 2\sqrt{2/3}, \qquad \ell_{32} = -\sqrt{3/2}, \qquad \ell_{33} = \sqrt{7/2}$$