MATH-4800 **NUMERICAL COMPUTING** **Summer 2023**

Assignment 3, due before class, Thursday June 15, 2023.

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1. Let

$$A = \begin{bmatrix} 1 & -2 & -1 \\ \alpha & 3 & 1 \\ -1 & \alpha & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \beta \\ -4 \\ 2 \end{bmatrix}$$

and let $\mathbf{x}$ be the solution of $A\mathbf{x} = \mathbf{b}$, if it exists. Determine conditions on the constants α and β such that (a) a solution or solutions exist and (b) a unique solution exists.

(a) $\det(A) = 1 \cdot (3-\alpha) - (-2)(\alpha+1) - 1 \cdot (\alpha^2+3)$

$= 3 - \alpha + 2\alpha + 2 - \alpha^2 - 3$

$= -\alpha^2 + \alpha + 2$

when $\det(A) \neq 0$, $-\alpha^2 + \alpha + 2 \neq 0$

$(\alpha-2)(\alpha+1) \neq 0$

$\alpha \neq -1, 2$

matrix A is non-singular, and thus, a solution exists.

when $\alpha = -1$, $\left(\begin{array}{ccc|c} 1 & -2 & -1 & \beta \\ -1 & 3 & 1 & -4 \\ -1 & -1 & 1 & 2 \end{array}\right) \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to R_1 + R_2}$

$\left(\begin{array}{ccc|c} 1 & -2 & -1 & \beta \\ 0 & 1 & 0 & \beta - 4 \\ 0 & -3 & 0 & \beta + 2 \end{array}\right) \xrightarrow{R_3 \to 3R_2 + R_3} \left(\begin{array}{ccc|c} 1 & -2 & -1 & \beta \\ 0 & 1 & 0 & \beta - 4 \\ 0 & 0 & 0 & 4\beta - 10 \end{array}\right)$

in order for solutions to exist, $4\beta - 10 = 0 \Rightarrow \beta = 5/2$.

when $\alpha = 2$, $\left(\begin{array}{ccc|c} 1 & -2 & -1 & \beta \\ 2 & 3 & 1 & -4 \\ -1 & 2 & 1 & 2 \end{array}\right) \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to R_2 - 2R_1}$

$\left(\begin{array}{ccc|c} 1 & -2 & -1 & \beta \\ 0 & 7 & 3 & -4-\beta \\ 0 & 0 & 0 & \beta+2 \end{array}\right)$ in order for solutions to exist,

$\beta + 2 = 0 \Rightarrow \beta = -2$.

Therefore, when $\boxed{\alpha \neq -1, 2 \text{ or } \alpha = -1, \beta = 5/2, \text{ or } \alpha = 2, \beta = -2}$, //

a solution or solutions exist.

(b) When $\boxed{\alpha \neq -1, 2}$, determinant of A is non-zero. It follows that matrix A is non-singular, which implies that A is invertible, and thus $A^{-1}b$ is a unique solution of $Ax = b$, regardless of the value of β we choose.

2. (*a*) Consider the two augmented matrices of the form $[A|\mathbf{b}]$ in text exercise 4 on page 81. For each matrix, use row operations to reduce it to the upper triangular form $[U|\mathbf{c}]$ and then find the solution of $U\mathbf{x} = \mathbf{c}$ using backwards substitution.

(*b*) Text exercise 6 and 7 on page 82.

(a) $$\left(\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 6 & -6 & 1 & 2 \\ -3 & 8 & 2 & -1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 0 & 2 & 5 & -4 \\ 0 & 4 & 0 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 0 & 2 & 5 & -4 \\ 0 & 0 & -10 & 10 \end{array}\right) = [\,U \mid c\,]$$

$$\left.\begin{aligned} -10x_3 &= 10 \\ 2x_2 + 5x_3 &= -4 \\ 3x_1 - 4x_2 - 2x_3 &= 3 \end{aligned}\right\} \quad \begin{aligned} & x_3 = -1 \\ & 2x_2 = -4 + 5 = 1 \Rightarrow x_2 = 1/2 \\ & 3x_1 = 3 + 2 - 2 = 3 \Rightarrow x_1 = 1 \end{aligned}$$

$$\boxed{\text{Solution } x = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}}$$

//

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 6 & 2 & -2 & 8 \\ 4 & 6 & -3 & 5 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 4 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 4R_1} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\left.\begin{aligned} 3x_3 &= 9 \\ -x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 2 \end{aligned}\right\} \quad \begin{aligned} & x_3 = 3 \\ & -x_2 = 2 - 3 = -1 \Rightarrow x_2 = 1 \\ & 2x_1 = 2 - 1 + 3 = 4 \Rightarrow x_1 = 2 \end{aligned}$$

$$\boxed{\text{Solution } x = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}} \; //$$

(b) Given that the computer takes 0.005 seconds to perform back-substitution which has complexity of $O(n^2)$.

Now as a full Gaussian elimination takes approximately $\frac{2}{3}n^3$ operations, with $n = 5000$, it follows that

$$n^2 \cdot \frac{2}{3}n$$

$$= 0.005 \cdot \frac{2 \cdot 5000}{3}$$

$$\approx \boxed{17 \text{ s}} \; //.$$

Now given the computer takes 0.002 seconds to perform back-substitution on a 4000 by 4000 matrix, it follows that it can perform

$$\frac{4000^2}{0.002} = 8 \times 10^9 \text{ operations/second}$$

Solving a general system of 9000 equations with 9000 unknowns will take approximately $9000^2 + \frac{2}{3} 9000^3 = 4.86081 \times 10^{11}$ operations, and the computer will need approximately $\frac{4.86081 \times 10^{11}}{8 \times 10^9} \approx \boxed{61 \text{ s}}$ to compute.

3. Let $\hat{A}$ be an $n \times (n+2)$ matrix and consider the following steps in a Matlab code:

```
for kb=1:2
  x(n,kb)=Ahat(n,n+kb)/Ahat(n,n);
  for i=n-1:-1:1
    sum=Ahat(i,n+kb);
    for j=i+1:n
      sum=sum-Ahat(i,j)*x(j,kb);
    end
    x(i,kb)=sum/Ahat(i,i);
  end
end
```

Determine the number of flops used to compute the elements of x for a given positive interger n. (Find an expression for the number of flops exactly, without approximation for large n.)

Divisions $\quad f_d = \sum_{kb=1}^{2} 1 + \sum_{kb=1}^{2} \sum_{i=1}^{n-1} 1$

$= 2 + 2(n-1)$

$= 2n$

Multiplications $\quad f_m = \sum_{kb=1}^{2} 1 \sum_{j=i+1}^{n} = \sum_{kb=1}^{2} 1 \sum_{i=1}^{n-1} (n-(i+1)+1)$

$= 2 \sum_{i=1}^{n-1} (n-i) = 2\left(\sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i \right)$

$= 2\left(n \sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} i \right)$

$= 2n(n-1) - 2 \cdot \frac{n(n-1)}{2}$

$= 2n^2 - 2n - n^2 + n$

$= n^2 - n$

Additions $\quad f_a = n^2 - n \quad$ same as multiplication

one floating-point subtraction from each multiplication

Total $= 2n + n^2 - n + n^2 - n = \boxed{2n^2 \text{ flops}}$ //

4. (a) Write a Matlab code to compute the LU-decomposition of a given $n \times n$ matrix A following the algorithm discussed in class. Your code need not perform row interchanges (pivoting), but should check for very small pivot elements and provide a warning if one is encountered.

(b) Use your code to compute the L and U factors for the following matrices:

(i) `A=[6 3 2; -1 4 2; 1 3 -5];` (ii) `A=hilb(5);`

Print out L and U and compute `norm(A-L*U)` for each case. The latter is Matlab's calculation of the distance (i.e. norm) between A and the product LU. What are the expected values for this norm?

```
function [L,U] = myLU(A)

[m,n]=size(A);
if m~=n
    error("Matrix is not square!")
end
L=eye(n);
U=A;

for k=1:n-1
    if abs(U(k,k))<10e-16
        printf("Warning: value less than 1-e-16\n");
    end

    for i=k+1:n
        L(i,k)=U(i,k)/U(k,k);
        for j=k:n
            U(i,j)=U(i,j)-L(i,k)*U(k,j);
        end
    end
end

% HW 3 Problem 4 part (i)
[L,U] = myLU([6 3 2; -1 4 2; 1 3 -5])
norm([6 3 2; -1 4 2; 1 3 -5]- L*U)
% HW 3 Problem 4 part (ii)
[L,U] = myLU(hilb(5))
norm(hilb(5)- L*U)
```

Part i

```
L =

    1.0000         0         0
   -0.1667    1.0000         0
    0.1667    0.5556    1.0000


U =

    6.0000    3.0000    2.0000
         0    4.5000    2.3333
         0         0   -6.6296


ans =

     0
```

Part ii

```
L =

    1.0000         0         0         0         0
    0.5000    1.0000         0         0         0
    0.3333    1.0000    1.0000         0         0
    0.2500    0.9000    1.5000    1.0000         0
    0.2000    0.8000    1.7143    2.0000    1.0000


U =

    1.0000    0.5000    0.3333    0.2500    0.2000
         0    0.0833    0.0833    0.0750    0.0667
         0   -0.0000    0.0056    0.0083    0.0095
         0         0         0    0.0004    0.0007
         0         0         0         0    0.0000


ans =

   4.4909e-17
```

5. (a) An $n \times n$ elimination matrix M has the form

$$M = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & -m_{k+1,k} & & & \\ & 0 & \vdots & & 0 & \ddots \\ & & -m_{n,k} & & & 1 \end{bmatrix} \quad \text{for some integer } k \in [1, n-1]$$

Find 4×4 elimination matrices M_1 and M_2 satisfying

$$M_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad M_2 \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

(b) An $n \times n$ permutation matrix P is a row or column permutation of the $n \times n$ identity matrix. Thus, P has exactly one 1 in every column and row. The following are examples of 3×3 permutation matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find 4×4 permutation matrices P_1 and P_2 satisfying

$$P_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \qquad P_2 \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \\ 2 \end{bmatrix}$$

(a) $2 - 2 \cdot 1 = 3 - 3 \cdot 1 = 4 - 4 \cdot 1 = 0$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$m_{2,1} = 2 \qquad m_{3,1} = 3 \qquad m_{4,1} = 4$

$$\boxed{M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}}$$

//

$$4 + 4 \cdot (-1) = -3 - 3 \cdot (-1) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$m_{3,2} = -4 \qquad\qquad m_{4,2} = 3$$

$$\boxed{M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}}$$

By definition given above, elimination matrix M has the form of $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$, where the bottom entries should be in the same column.

(b) $$\boxed{P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

For permutation matrix P, the product PA is a new matrix whose rows consists of the rows of A rearranged in the new order.