MATH 2400 EXAM 3 SPRING 2023 Name Haowen He Section

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. (1.5 pts) Find the inverse Laplace Transform of $F(s) = \frac{4-2s}{s^2+2s+10}$. $= \frac{-2(s+1)+6}{(s+1)^2+9}$

a. $f(t) = 2e^{t}(\sin 3t - \cos 3t)$

b. $f(t) = 2e^{-t}(\sin 3t - \cos 3t)$ $\xrightarrow{\mathcal{L}^{-1}}$ $-2e^{-t}\cos 3t + 2e^{-t}\sin 3t$

c. $f(t) = 2e^{3t}(\sin t - \cos t)$

d. $f(t) = 2e^{-3t}(\sin t - \cos t)$

answer: b

2. (1.5 pts) Given the system of two species, *x* and *y*, in a closed environment,

$$\frac{dx}{dt} = x\left(1 - \frac{y}{3}\right); \qquad \frac{dy}{dt} = y\left(-1 + \frac{x}{3}\right)$$

Which of the following is true?

a. *x* is the prey, *y* is the predator

b. *x* is the predator, *y* is the prey

c. *x* and *y* are competing species

d. None of the above

answer: a

3. (1.5 pts) Given the Fourier series of a periodic function *f* (*x*) be

$$f(x) = \frac{2}{5} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1 + \cos(n\pi))\sin(n\pi x)$$

Find a_0, a_1, b_3

a. ~~$a_0 = \frac{1}{5}$~~; $a_1 = 0$; $b_3 = \frac{2}{3\pi}$

b. ~~$a_0 = \frac{1}{5}$~~; $a_1 = \frac{2}{\pi}$; $b_3 = 0$

c. $a_0 = \frac{4}{5}$; $a_1 = \frac{2}{\pi}$; $b_3 = \frac{2}{3\pi}$

d. $a_0 = \frac{4}{5}$; $a_1 = 0$; $b_3 = 0$

$\frac{a_0}{2} = \frac{2}{5}$

$a_0 = \frac{4}{5}$

$b_3 = \frac{1}{3}(1 + \cancel{\cos 3\pi})$ -1

$= 0$

answer: d

4. **(5.5 pts)** Given the nonlinear system describing two species x and y who are competing for food in a closed environment:

$$x' = x(1 - x - y) = F\ ; \qquad y' = y\left(\frac{3}{4} - y - \frac{x}{2}\right) = G$$

a. **(5 pts)** Find all the steady state solutions, their eigenvalues and stabilities. (Hint: use Jacobian to find eigenvalues)

b. **(0.5 pt)** Do the two species (x and y) co-exist? (Yes ✓ / No)

$$A = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ -y/2 & \frac{3}{4}-2y-\frac{x}{2} \end{pmatrix}$$

a) $x' = 0 \Rightarrow x = 0$ OR $x = 1 - y$

$x = 0,$

$y' = y\left(\frac{3}{4} - y\right) = 0$

$\Rightarrow y = 0$ OR $y = \frac{3}{4}$

$(0, 0)$, $\left(0, \frac{3}{4}\right)$

$x = 1 - y,$

$y' = y\left(\frac{3}{4} - y - \frac{1}{2}(1-y)\right) = 0$

$y = 0$ OR $\frac{1}{4} - \frac{1}{2}y = 0$

$x = 1,$ $\quad y = \frac{1}{2}$

$(1, 0)$ $\quad x = 1 - \frac{1}{2} = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{1}{2}\right)$

@ $(0,0)$, $A\big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix} \Rightarrow \lambda = 1, \frac{3}{4}$ (unstable)

@ $\left(0, \frac{3}{4}\right)$, $A\big|_{(0,\frac{3}{4})} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix} \Rightarrow \lambda = \frac{1}{4}, -\frac{3}{4}$ (unstable)

@ $(1,0)$, $A\big|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix} \Rightarrow \lambda = -1, \frac{1}{4}$ (unstable)

@ $\left(\frac{1}{2}, \frac{1}{2}\right)$, $A\big|_{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \Rightarrow \left(-\frac{1}{2} - \lambda\right)^2 + \frac{1}{8} = 0$

$\lambda^2 + \lambda + \frac{1}{4} + \frac{1}{8} = 0$

$\lambda = \dfrac{-1 \pm \sqrt{1 - 4\left(\frac{3}{8}\right)}}{2} = -\frac{1}{2} \pm \frac{1}{2\sqrt{2}}$

(asymptotically stable)

b) Yes.

5. (5 pts) Use the method of Laplace transforms to solve an IVP

$\mathcal{L}\left(y'' + 3y' + 2y = u_\pi(t) + 2\delta(t-\pi)\right)$, where $y(0) = 0$; $y'(0) = 1$.

$$s^2\mathcal{L}\{y\} - y'(0) - sy(0) + 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s} + 2e^{-\pi s}$$

$$\mathcal{L}\{y\}(s^2 + 3s + 2) = \frac{1+2s}{s}e^{-\pi s} + 1$$

$$\mathcal{L}\{y\} = \frac{1+2s}{s(s+2)(s+1)}e^{-\pi s} + \frac{1}{(s+2)(s+1)}$$

Consider $\frac{1+2s}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \underbrace{\frac{1}{2s} - \frac{3}{2(s+2)} + \frac{1}{s+1}}_{F_1(s)}$

$$= \frac{A(s+2)(s+1) + Bs(s+1) + Cs(s+2)}{s(s+2)(s+1)}$$

s^2: $A + B + C = 0$ — (1)

s^1: $3A + B + 2C = 2$ — (2)

s^0: $2A = 1 \Rightarrow A = \frac{1}{2}$

(2) − (1),

$2A + C = 2$

$C = 1$

$B = -\frac{3}{2}$

Consider $\frac{1}{(s+2)(s+1)} = \frac{D}{s+2} + \frac{E}{s+1} = \frac{Ds + D + Es + 2E}{(s+2)(s+1)}$

$\Downarrow$

$= -\frac{1}{s+2} + \frac{1}{s+1}$

s^1: $D + E = 0$

s^0: $D + 2E = 1$

$\Rightarrow E = 1,\ D = -1$

$$F_1(s) \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2} - \frac{3}{2}e^{-2t} + e^{-t} = f(t)$$

$$f(t-\pi) = \frac{1}{2} - \frac{3}{2}e^{-2(t-\pi)} + e^{-(t-\pi)}$$

$$\boxed{y = u_\pi(t)\left(\frac{1}{2} - \frac{3}{2}e^{-2(t-\pi)} + e^{-(t-\pi)}\right) - e^{-2t} + e^{-t}}$$

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6. (5 pts) Find the Fourier Series for $f(x) = \begin{cases} -x & -1 \le x < 0 \\ 0 & 0 \le x < 1 \end{cases}$; $f(x+2) = f(x)$.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \int_{-1}^{0} -x\,dx = -\left(\frac{x^2}{2}\right)\Big|_{-1}^{0} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \int_{-1}^{0} -x\cos n\pi x\,dx$$

let $u = x$ $\quad dv = \cos n\pi x\,dx$

$du = dx$ $\quad v = \frac{1}{n\pi}\sin n\pi x$

$$= -\left(\frac{x}{n\pi}\sin n\pi x\Big|_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0}\sin n\pi x\,dx\right)$$

$$= -\frac{1}{n^2\pi^2}\cos n\pi x\Big|_{-1}^{0} = -\frac{1}{n^2\pi^2}(1-\cos n\pi)$$

$$= \frac{1}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0, & n = 2k \\ -\frac{2}{n^2\pi^2}, & n = 2k \pm 1 \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \int_{-1}^{0} -x\sin n\pi x\,dx$$

let $u = x$ $\quad dv = \sin n\pi x\,dx$

$du = dx$ $\quad v = -\frac{1}{n\pi}\cos n\pi x$

$$= -\left(-\frac{x}{n\pi}\cos n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos n\pi x\,dx\right)$$

$$= \frac{1}{n\pi}\cos n\pi - \cancel{\frac{1}{n^2\pi^2}\sin n\pi x\Big|_{-1}^{0}}^{\,0} = \begin{cases} \frac{1}{n\pi}, & n = 2k \\ -\frac{1}{n\pi}, & n = 2k \pm 1 \end{cases}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}(\cos n\pi - 1)\cos n\pi x + \frac{1}{n\pi}\cos n\pi \sin n\pi x\right)$$

OR

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{-2}{(2n-1)^2\pi^2}\cos(2n-1)\pi x + \frac{(-1)^n}{n\pi}\sin n\pi x\right)$$

Bonus Question (1 pt) Find the Laplace Transform of $f(t) = \cos(t - \frac{\pi}{2})u_\pi(t)$.

$g(t-\pi) = \cos(t - \frac{\pi}{2})$

$g(t) = \cos(t + \frac{\pi}{2}) = -\sin t \xrightarrow{\mathcal{L}} G(s) = -\frac{1}{s^2+1}$

$f(t) \xrightarrow{\mathcal{L}} \frac{-e^{-\pi s}}{s^2+1}$

Useful Formulas:

1. Fourier Series $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

2. Laplace Transform for derivatives:

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - y'(0) - sy(0)$$

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - y^{(n-1)}(0) - sy^{(n-2)}(0) - \cdots - s^{n-1}y(0)$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
5. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
6. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
7. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
9. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
10. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
11. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
12. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
13. $e^{ct}f(t)$	$F(s-c)$
14. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
15. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
16. $\delta(t-c)$	e^{-cs}
17. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18. $(-t)^n f(t)$	$F^{(n)}(s)$