MATH 2400 EXAM 1 SPRING 2023 Name Haowen He Section

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. **(1.5 pts)** Given $y = y(x)$. Which of the following methods can be used to solve $x^2y' + xy = 1/x$?

first-order ODE

separable variable — integrating factor

$\frac{dy}{dx} = \frac{1}{x^3} - \frac{y}{x}$

$y' + \frac{1}{x}y = \frac{1}{x^3}$

$\mu = e^{\int 1/x\, dx}$

   a. Integrating factor
   b. Separable variables X
   c. ~~Undetermined Coefficients~~
   d. ~~Variation of Parameters~~

answer: a

2. **(1.5 pts)** Which of the following pairs is linearly dependent?

   a. $3e^x, xe^x$
   b. $x \sin x$, $2x / \csc x$ $2x\sin x$
   c. $3xe^{2x}, 2xe^{3x}$ $e^{3x} = e^{2x} \cdot e^x$
   d. $x \cos x$, $x \sec x = \frac{x}{\cos x}$

answer: b

3. **(1.5 pts)** Which of the following equations is second order, linear, non-homogeneous ODE? Given $y = y(t)$.

   a. $y'' + 3y' - \cos y = 0$ X
   b. $y'' + 3y' + \cos t^2 = 0$
   c. $y'' + 3y' = e^{2t}y$ X
   d. $y'' + 3y' - y = 0$ X

answer: b

4. (5pts) Find the particular solution of $(1+x^2)y' = (2+y)x$ by any method, where $y(0) = 0$.

$$\frac{dy}{dx} = \frac{(2+y)x}{1+x^2}$$

$$\Rightarrow \int \frac{x}{1+x^2}\,dx$$

let $u = 1+x^2$

$du = 2x\,dx$

$$\int \frac{dy}{2+y} = \int \frac{x}{1+x^2}\,dx$$

$$\frac{1}{2}\int \frac{du}{u}$$

$$\ln|2+y| = \frac{1}{2}\ln|1+x^2| + c$$

$$2+y = e^{\ln|1+x^2|^{1/2}+c}$$

$$= \frac{1}{2}\ln|u|$$

$$y = c_1|1+x^2|^{1/2} - 2$$

$$y(0) = 0 \Rightarrow c_1 - 2 = 0 \Rightarrow c_1 = 2$$

$$\boxed{y = 2\left[|1+x^2|^{1/2} - 1\right]}$$

5. **(5.5 pts)** Given $y = y(x)$. Find the general solutions of $y'' + 4y = \csc 2x$ by using ***variation of parameters***.

solve $y_h'' + 4y_h = 0$

let $y_h = e^{rx}$

$\Rightarrow r^2 + 4 = 0$

$r = \pm 2i \Rightarrow y_h = C_1\cos 2x + C_2\sin 2x$

$y_1 = \cos 2x, \qquad y_2 = \sin 2x,$

$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ \frac{g(x)}{a_2} & y_2' \end{vmatrix} = -y_2\left(g(x)/a_2\right) = -\sin 2x(\csc 2x) = -1$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x)/a_2 \end{vmatrix} = y_1\left(\frac{g(x)}{a_2}\right) = \cos 2x\left(\csc 2x\right) = \frac{\cos 2x}{\sin 2x}$$

$$u_1' = \frac{W_1}{W} = -\frac{1}{2} \Rightarrow u_1 = -\frac{1}{2}x$$

$$u_2' = \frac{W_2}{W} = \frac{1}{2}\frac{\cos 2x}{\sin 2x} \Rightarrow u_2 = \frac{1}{2}\int \frac{\cos 2x}{\sin 2x}dx \qquad \text{let } v = \sin 2x$$

$$dv = 2\cos 2x\,dx$$

$$= \frac{1}{4}\int \frac{dv}{v} = \frac{1}{4}\ln|\sin 2x|$$

$$\boxed{y = C_1\cos 2x + C_2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\ln|\sin 2x|}$$

6. **(5 pts)** By using ***method of undetermined coefficients*** or other methods, find the general solution for the 2nd order differential equation $y'' + 3y' + 2y = 4e^{2x} + 5$.

solve $y_h'' + 3y_h' + 2y_h = 0$ let $y_h = e^{rx}$

$\Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow r = -1, -2$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

$g(x) = 4e^{2x} + 5$

let $y_p = Ae^{2x} + B$

$y_p' = 2Ae^{2x}$ $\Rightarrow y_p'' + 3y_p' + 2y_p = 4e^{2x} + 5$

$y_p'' = 4Ae^{2x}$

e^{2x}: $4A + 3(2A) + 2A = 4 \Rightarrow 12A = 4 \Rightarrow A = \frac{1}{3}$

constant: $2B = 5 \Rightarrow B = \frac{5}{2}$

$\Rightarrow y = y_h + y_p = \boxed{c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{3}e^{2x} + \frac{5}{2}}$ //

Bonus Question (1 pt) Given $y'' + 3y' - y = 2, y(0) = 1;\ y'(0) = 2.$ Find $y'''(0)$.

(Hint: start with finding $y''(0)$)

$$y''(0) + 3\cancelto{2}{y'(0)} - \cancelto{1}{y(0)} = 2$$

$$y''(0) = -3$$

$$y'''(0) + 3y''(0) - y'(0) = 0$$

$$y'''(0) = y'(0) - 3y''(0)$$

$$= 2 - 3 \cdot (-3)$$

$$= 11$$