MATH 2400 EXAM 1 SPRING 2023 Name Haowen He Section ______

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. **(1.5 pts)** Given $y = y(x)$. Which of the following methods can be used to solve $x^2y' + xy = 1/x$?

   first-order ODE

   separable variable — integrating factor

   a. Integrating factor
   b. Separable variables X
   c. ~~Undetermined Coefficients~~
   d. ~~Variation of Parameters~~

   $\frac{dy}{dx} = \frac{1}{x^3} - \frac{y}{x}$

   $y' + \frac{1}{x}y = \frac{1}{x^3}$

   $\mu = e^{\int 1/x\,dx}$

   answer: a

2. **(1.5 pts)** Which of the following pairs is linearly dependent?

   a. $3e^x, xe^x$
   b. $x\sin x$, $2x/\csc x$ $= 2x\sin x$
   c. $3xe^{2x}$, $2xe^{3x}$ $e^{3x} = e^{2x} \cdot e^{x}$
   d. $x\cos x$, $x\sec x = \frac{x}{\cos x}$

   answer: b

3. **(1.5 pts)** Which of the following equations is second order, linear, non-homogeneous ODE? Given $y = y(t)$.

   a. $y'' + 3y' - \cos y = 0$ X
   b. $y'' + 3y' + \cos t^2 = 0$
   c. $y'' + 3y' = e^{2t}y$ X
   d. $y'' + 3y' - y = 0$ X

   answer: b

4. (5pts) Find the particular solution of $(1+x^2)y' = (2+y)x$ by any method, where $y(0) = 0$.

$$\frac{dy}{dx} = \frac{(2+y)x}{1+x^2}$$

$$\int \frac{dy}{2+y} = \int \frac{x}{1+x^2}\,dx$$

$$\ln|2+y| = \frac{1}{2}\ln|1+x^2| + C$$

$$2+y = e^{\ln|1+x^2|^{1/2}+C}$$

$$y = C_1|1+x^2|^{1/2} - 2$$

$$\Rightarrow \int \frac{x}{1+x^2}\,dx$$

let $u = 1+x^2$

$du = 2x\,dx$

$$\frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|$$

$$y(0) = 0 \Rightarrow C_1 - 2 = 0 \Rightarrow C_1 = 2$$

$$\boxed{y = 2\left[|1+x^2|^{1/2} - 1\right]}$$

//

5. (5.5 pts) Given $y = y(x)$. Find the general solutions of $y'' + 4y = \csc 2x$ by using ***variation of parameters***.

solve $y_h'' + 4y_h = 0$

let $y_h = e^{rx}$

$\Rightarrow r^2 + 4 = 0$

$r = \pm 2i \Rightarrow y_h = c_1\cos 2x + c_2 \sin 2x$

$y_1 = \cos 2x,\qquad y_2 = \sin 2x,$

$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ \frac{g(x)}{a_2} & y_2' \end{vmatrix} = -y_2\left(g(x)/a_2\right) = -\sin 2x(\csc 2x) = -1$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x)/a_2 \end{vmatrix} = y_1\left(\frac{g(x)}{a_2}\right) = \cos 2x\left(\csc 2x\right) = \frac{\cos 2x}{\sin 2x}$$

$$u_1' = \frac{W_1}{W} = -\frac{1}{2} \Rightarrow u_1 = -\frac{1}{2}x$$

$$u_2' = \frac{W_2}{W} = \frac{1}{2}\frac{\cos 2x}{\sin 2x} \Rightarrow u_2 = \frac{1}{2}\int \frac{\cos 2x}{\sin 2x}\,dx \qquad \text{let } v = \sin 2x$$

$$dv = 2\cos 2x\,dx$$

$$= \frac{1}{4}\int \frac{dv}{v} = \frac{1}{4}\ln|\sin 2x|$$

$$\boxed{y = c_1\cos 2x + c_2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \ln|\sin 2x|}$$

//

6. (5 pts) By using ***method of undetermined coefficients*** or other methods, find the general solution for the 2nd order differential equation $y'' + 3y' + 2y = 4e^{2x} + 5$.

solve $y_h'' + 3y_h' + 2y_h = 0$ let $y_h = e^{rx}$

$\Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow r = -1, -2$

$y_h = C_1 e^{-x} + C_2 e^{-2x}$

$g(x) = 4e^{2x} + 5$

let $y_p = Ae^{2x} + B$

$y_p' = 2Ae^{2x}$ $\Rightarrow y_p'' + 3y_p' + 2y_p = 4e^{2x} + 5$

$y_p'' = 4Ae^{2x}$

e^{2x}: $4A + 3(2A) + 2A = 4 \Rightarrow 12A = 4 \Rightarrow A = \frac{1}{3}$

constant: $2B = 5 \Rightarrow B = \frac{5}{2}$

$\Rightarrow y = y_h + y_p = \boxed{C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{3}e^{2x} + \frac{5}{2}}$ //

Bonus Question (1 pt) Given $y" + 3y' - y = 2, y(0) = 1;\ y'(0) = 2.$ Find $y'''(0)$.

(Hint: start with finding $y''(0)$)

$$y''(0) + 3\cancelto{2}{y'(0)} - \cancelto{1}{y(0)} = 2$$

$$y''(0) = -3$$

$$y'''(0) + 3y''(0) - y'(0) = 0$$

$$y'''(0) = y'(0) - 3y''(0)$$

$$= 2 - 3 \cdot (-3)$$

$$= 11$$

MATH 2400 EXAM 2 SPRING 2023 Name Haowen He Section

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. **(1.5 pts)** Given A = $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$. Which of the following statement is true?

   a. A is defective.
   b. A is chaotic.
   c. A is singular.
   d. None of the above.

   $\det(A) = 1\cdot(-2) - (-1)\cdot 2 = 0$

   answer: c

2. **(1.5 pts)** Which of the following pair of vectors is linearly independent? Given *a, b* are arbitrary constants.

   a. $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} 3\pi \\ 6\pi \end{pmatrix}$
   b. $\begin{pmatrix} ab \\ 2a \end{pmatrix}, \begin{pmatrix} ab^2 \\ 2ab \end{pmatrix}$
   c. $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} 3b \\ b \end{pmatrix}$
   d. $\begin{pmatrix} 2\pi \\ \pi e \end{pmatrix}, \begin{pmatrix} 2 \\ e \end{pmatrix}$

   answer: c

3. **(1.5 pts)** Given the solution of a linear system be $\boldsymbol{y} = \begin{pmatrix} 3C_1e^{2t} + 2C_2e^{-t} \\ C_1e^{2t} - C_2e^{-t} \end{pmatrix}$. This system is

   a. Asymptotically stable
   b. Unstable
   c. Neutrally stable
   d. None of the above

   $\lambda_1 = 2$ & $\lambda_2 = -1$

   saddle point

   $\Downarrow$

   unstable

   answer: b

4. **(5 pts)** Find the general solution of $2x^2y'' + 3xy' - y = x$, $y_1 = x^{1/2}$ by ***reduction of order***.

let $y = y_1 v = x^{1/2} v$

$$y' = \frac{1}{2}x^{-1/2}v + x^{1/2}v'$$

$$y'' = -\frac{1}{4}x^{-3/2}v + x^{-1/2}v' + x^{1/2}v''$$

equation becomes

$$2x^2\left(\cancel{-\frac{1}{4}x^{-3/2}v} + x^{-1/2}v' + x^{1/2}v''\right) + 3x\left(\cancel{\frac{1}{2}x^{-1/2}v} + x^{1/2}v'\right) - \cancel{x^{1/2}v} = x$$

$$2x^{5/2}v'' + 5x^{3/2}v' = x$$

let $u = v'$

$$u' + \frac{5}{2}x^{-1}u = \frac{1}{2}x^{-3/2}$$

let $\mu = e^{\frac{5}{2}\int 1/x \, dx}$

$$= x^{5/2}$$

$$(x^{5/2}u)' = x^{5/2} \cdot \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2}x$$

$$x^{5/2}u = \frac{1}{4}x^2 + c_1$$

$$u = \frac{1}{4}x^{-1/2} + c_1 x^{-5/2}$$

$$v = \int u \, dx$$

$$= \frac{1}{2}x^{1/2} \cancel{- \frac{2}{3}c_1}^{\,c_3} x^{-3/2} + c_2$$

$$y = x^{1/2}v$$

$$= \boxed{\frac{1}{2}x - c_3 x^{-1} + c_2 x^{1/2}}$$

5. (5.5 pts) Given a system $\boldsymbol{y}' = \begin{pmatrix} 3 & -4 \\ 0 & 3 \end{pmatrix} \boldsymbol{y}$, with the matrix $A = \begin{pmatrix} 3 & -4 \\ 0 & 3 \end{pmatrix}$ being defective.

a. (2 pts) Find the corresponding eigenvalues , λ, and the first eigenvector, $\boldsymbol{x}^{(1)}$.
b.(3.5 pts) Assume $\boldsymbol{y}^{(2)} = \boldsymbol{x}^{(2)} e^{\lambda t}$, where $\boldsymbol{x}^{(2)} = t\boldsymbol{x}^{(1)} + \boldsymbol{\varphi}$ and $(A - \lambda I)\boldsymbol{\varphi} = \boldsymbol{x}^{(1)}$. Find the particular solution of the system, where $\boldsymbol{y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

a) $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -4 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 3$ (repeated)

$\lambda = 3$, $(A - 3I)\vec{x} = 0$

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = a$; $x_2 = 0$

$$\vec{x} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b) $(A - 3I)\vec{\varphi} = \vec{x}^{(1)}$

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\varphi_1 = a$; $\varphi_2 = -\frac{1}{4}$

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} a \\ -1/4 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$$

$$\vec{y} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}$$

$$+ c_2\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}\right) e^{3t}$$

$$= \begin{pmatrix} c_3 + c_2 t \\ -\frac{1}{4}c_2 \end{pmatrix} e^{3t}$$

$$\vec{y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} c_3 \\ -\frac{1}{4}c_2 \end{pmatrix}$$

$\Rightarrow$ $c_2 = -12$;

$c_3 = 2$

$$\Rightarrow \boxed{\vec{y} = \begin{pmatrix} 2 - 12t \\ 3 \end{pmatrix} e^{3t}}$$ //

6. (5 pts) Given the nonlinear system:

$$\frac{dx}{dt} = -(2+y)(x+y) = F \qquad \frac{dy}{dt} = -y(1-x) = G$$

a. (3 pts) Find the 3 steady state solutions (i.e. 3 critical points).

b. (2 pts) Find the Jacobian (matrix A) of the system.

a) $x' = F = 0 \Rightarrow y = -2$ OR $y = -x$

$y' = G = 0$

when $y = -2$

$y' = 2(1-x) = 0$

$x = 1$

$\boxed{(1, -2)}$

when $y = -x$

$y' = x(1-x) = 0$

$x = 0, 1$

$x = 0, \ y = 0, \ \boxed{(0,0)}$

$x = 1, \ y = -1, \ \boxed{(1,-1)}$

b) $A = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$

$= \begin{pmatrix} -2-y & -x-2-2y \\ y & -1+x \end{pmatrix}$

$F_x = -(2+y)$

$F_y = -(x+y) - (2+y)$

$G_x = y$

$G_y = -(1-x)$

7. Bonus Question (1 pt) Given a system $\boldsymbol{y}' = \begin{pmatrix} \alpha+1 & -3 \\ 2 & \alpha+1 \end{pmatrix} \boldsymbol{y}$. Determine the value(s) of α where the system is ***neutrally stable***.

$$|A - \lambda I| = 0$$

$$(\alpha + 1 - \lambda)^2 + 6 = 0$$

$$\alpha + 1 - \lambda = \pm\sqrt{6}\, i$$

$$\lambda = \alpha + 1 \mp \sqrt{6}\, i$$

neutrally stable

$$\alpha + 1 = 0$$

$$\alpha = -1$$

MATH 2400 EXAM 3 SPRING 2023 Name Haowen He Section

(3 Multiple Choice Questions) 4.5pts + (3 Questions) 15.5pts + (1 Question) 1 bonus pt

Multiple Choice Questions

1. (1.5 pts) Find the inverse Laplace Transform of $F(s) = \frac{4-2s}{s^2+2s+10}$. $= \frac{-2(s+1)+6}{(s+1)^2+9}$

a. $f(t) = 2e^{t}(\sin 3t - \cos 3t)$

b. $f(t) = 2e^{-t}(\sin 3t - \cos 3t)$ $\xrightarrow{\mathcal{L}^{-1}}$ $-2e^{-t}\cos 3t + 2e^{-t}\sin 3t$

c. $f(t) = 2e^{3t}(\sin t - \cos t)$

d. $f(t) = 2e^{-3t}(\sin t - \cos t)$ answer: b

2. (1.5 pts) Given the system of two species, x and y, in a closed environment,

$$\frac{dx}{dt} = x\left(1 - \frac{y}{3}\right); \qquad \frac{dy}{dt} = y\left(-1 + \frac{x}{3}\right)$$

Which of the following is true?

a. x is the prey, y is the predator

b. x is the predator, y is the prey

c. x and y are competing species

d. None of the above answer: a

3. (1.5 pts) Given the Fourier series of a periodic function $f(x)$ be

$$f(x) = \frac{2}{5} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(1 + \cos(n\pi))\sin(n\pi x)$$

Find a_0, a_1, b_3

a. $a_0 = \frac{1}{5}; a_1 = 0; b_3 = \frac{2}{3\pi}$

b. $a_0 = \frac{1}{5}; a_1 = \frac{2}{\pi}; b_3 = 0$

c. $a_0 = \frac{4}{5}; a_1 = \frac{2}{\pi}; b_3 = \frac{2}{3\pi}$

d. $a_0 = \frac{4}{5}; a_1 = 0; b_3 = 0$ answer: d

$\frac{a_0}{2} = \frac{2}{5}$

$a_0 = \frac{4}{5}$

$b_3 = \frac{1}{3}(1 + \cancel{\cos 3\pi}^{\,-1})$

$= 0$

4. **(5.5 pts)** Given the nonlinear system describing two species x and y who are competing for food in a closed environment:

$$x' = x(1 - x - y) = F \;; \qquad y' = y\left(\frac{3}{4} - y - \frac{x}{2}\right) = G$$

a. **(5 pts)** Find all the steady state solutions, their eigenvalues and stabilities. (Hint: use Jacobian to find eigenvalues)

b. **(0.5 pt)** Do the two species (x and y) co-exist? (Yes ✓ / No)

$$A = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ -y/2 & \frac{3}{4}-2y-\frac{x}{2} \end{pmatrix}$$

a) $x' = 0 \Rightarrow x = 0$ OR $x = 1-y$

$x = 0$,

$y' = y\left(\frac{3}{4} - y\right) = 0$

$\Rightarrow y = 0$ OR $y = \frac{3}{4}$

$(0,0)$, $\left(0, \frac{3}{4}\right)$

$x = 1-y$,

$y' = y\left(\frac{3}{4} - y - \frac{1}{2}(1-y)\right) = 0$

$y = 0$ OR $\frac{1}{4} - \frac{1}{2}y = 0$

$x = 1$, $(1, 0)$

$y = \frac{1}{2}$

$x = 1 - \frac{1}{2} = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{1}{2}\right)$

@ $(0,0)$, $A\big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix} \Rightarrow \lambda = 1, \frac{3}{4}$ (unstable)

@ $\left(0, \frac{3}{4}\right)$, $A\big|_{(0,\frac{3}{4})} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix} \Rightarrow \lambda = \frac{1}{4}, -\frac{3}{4}$ (unstable)

@ $(1,0)$, $A\big|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix} \Rightarrow \lambda = -1, \frac{1}{4}$ (unstable)

@ $\left(\frac{1}{2}, \frac{1}{2}\right)$, $A\big|_{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \Rightarrow \left(-\frac{1}{2} - \lambda\right)^2 + \frac{1}{8} = 0$

$\lambda^2 + \lambda + \frac{1}{4} + \frac{1}{8} = 0$

$\lambda = \dfrac{-1 \pm \sqrt{1 - 4\left(\frac{3}{8}\right)}}{2} = -\frac{1}{2} \pm \frac{1}{2\sqrt{2}}$

(asymptotically stable)

b) Yes.

5. (5 pts) Use the method of Laplace transforms to solve an IVP

$\mathcal{L}\left(y'' + 3y' + 2y = u_\pi(t) + 2\delta(t-\pi)\right)$ where $y(0) = 0;\quad y'(0) = 1.$

$$s^2\mathcal{L}\{y\} - y'(0) - sy(0) + 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s} + 2e^{-\pi s}$$

$$\mathcal{L}\{y\}(s^2+3s+2) = \frac{1+2s}{s}e^{-\pi s} + 1$$

$$\mathcal{L}\{y\} = \frac{1+2s}{s(s+2)(s+1)}e^{-\pi s} + \frac{1}{(s+2)(s+1)}$$

Consider $\frac{1+2s}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \underbrace{\frac{1}{2s} - \frac{3}{2(s+2)} + \frac{1}{s+1}}_{f_1(s)}$

$$= \frac{A(s+2)(s+1) + Bs(s+1) + Cs(s+2)}{s(s+2)(s+1)}$$

$s^2:\quad A + B + C = 0 \quad -(1)$

$s^1:\quad 3A + B + 2C = 2 \quad -(2)$

$s^0:\quad 2A = 1 \Rightarrow A = \frac{1}{2}$

$(2) - (1),$

$2A + C = 2$

$C = 1$

$B = -\frac{3}{2}$

Consider $\frac{1}{(s+2)(s+1)} = \frac{D}{s+2} + \frac{E}{s+1} = \frac{Ds + D + Es + 2E}{(s+2)(s+1)}$

$\Downarrow$

$= -\frac{1}{s+2} + \frac{1}{s+1}$

$s^1:\quad D + E = 0$

$s^0:\quad D + 2E = 1$

$\Rightarrow E = 1,\ D = -1$

$$f_1(s) \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2} - \frac{3}{2}e^{-2t} + e^{-t} = f(t)$$

$$f(t-\pi) = \frac{1}{2} - \frac{3}{2}e^{-2(t-\pi)} + e^{-(t-\pi)}$$

$$\boxed{y = u_\pi(t)\left(\frac{1}{2} - \frac{3}{2}e^{-2(t-\pi)} + e^{-(t-\pi)}\right) - e^{-2t} + e^{-t}}$$

//

6. (5 pts) Find the Fourier Series for $f(x) = \begin{cases} -x & -1 \le x < 0 \\ 0 & 0 \le x < 1 \end{cases}; f(x+2) = f(x)$.

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx = \int_{-1}^{0} -x\,dx = -\left(\frac{x^2}{2}\right)\Big|_{-1}^{0} = \frac{1}{2}$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \int_{-1}^{0} -x\cos n\pi x\,dx$$

let $u = x$ $\quad dv = \cos n\pi x\,dx$

$du = dx$ $\quad v = \frac{1}{n\pi}\sin n\pi x$

$$= -\left(\frac{x}{n\pi}\sin n\pi x\Big|_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0}\sin n\pi x\,dx\right)$$

$$= -\frac{1}{n^2\pi^2}\cos n\pi x\Big|_{-1}^{0} = -\frac{1}{n^2\pi^2}(1 - \cos n\pi)$$

$$= \frac{1}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0, & n = 2k \\ -\frac{2}{n^2\pi^2}, & n = 2k \pm 1 \end{cases}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \int_{-1}^{0} -x\sin n\pi x\,dx$$

let $u = x$ $\quad dv = \sin n\pi x\,dx$

$du = dx$ $\quad v = -\frac{1}{n\pi}\cos n\pi x$

$$= -\left(-\frac{x}{n\pi}\cos n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos n\pi x\,dx\right)$$

$$= \frac{1}{n\pi}\cos n\pi - \cancelto{0}{\frac{1}{n^2\pi^2}\sin n\pi x\Big|_{-1}^{0}} = \begin{cases} \frac{1}{n\pi}, & n = 2k \\ -\frac{1}{n\pi}, & n = 2k \pm 1 \end{cases}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}(\cos n\pi - 1)\cos n\pi x + \frac{1}{n\pi}\cos n\pi\sin n\pi x\right)$$

OR

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{-2}{(2n-1)^2\pi^2}\cos(2n-1)\pi x + \frac{(-1)^n}{n\pi}\sin n\pi x\right)$$

Bonus Question (1 pt) Find the Laplace Transform of $f(t) = \cos(t - \frac{\pi}{2})u_\pi(t)$.

$g(t-\pi) = \cos(t - \frac{\pi}{2})$

$g(t) = \cos(t + \frac{\pi}{2}) = -\sin t \xrightarrow{\mathcal{L}} G(s) = -\frac{1}{s^2+1}$

$f(t) \xrightarrow{\mathcal{L}} \frac{-e^{-\pi s}}{s^2+1}$

Useful Formulas:

1. Fourier Series $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

2. Laplace Transform for derivatives:

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - y'(0) - sy(0)$$

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - y^{(n-1)}(0) - sy^{(n-2)}(0) - \cdots - s^{n-1}y(0)$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
5. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
6. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$
7. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$
8. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
9. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
10. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
11. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
12. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
13. $e^{ct}f(t)$	$F(s-c)$
14. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
15. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
16. $\delta(t-c)$	e^{-cs}
17. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18. $(-t)^n f(t)$	$F^{(n)}(s)$